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The onset of cellular convection in a shallow two-dimensional container of fluid heated non-uniformly from below

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A theoretical study is made of the onset of cellular convection in a shallow two-dimensional container of fluid when the temperature difference between the horizontal boundaries is a monotonic function of horizontal distance. The typical lengthscale of the horizontal variation in the temperature difference is taken to be of the same order of magnitude as the length of the container, and both are much longer than the depth of the container. The resulting flow may be regarded as consisting of two parts, a steady base flow and a disturbed flow. It is found that a weak disturbance taking the form of transverse rolls is first set up near the endwalls, but as the temperature difference between the horizontal boundaries is uniformly increased, an instability in the form of longitudinal rolls takes place near the hotter end of the container. This description is in good qualitative agreement with experiment.

1. Introduction

In an earlier paper (Walton 1982, hereinafter referred to as I) we discussed the stability of a layer of fluid heated from below in which the depth of the layer increased monotonically with horizontal distance. Weakly nonlinear theory was used to determine the solution in a region, referred to as the 'transition' region, where the local Rayleigh number is close to the critical value for the onset of Rayleigh–Bénard convection in an unbounded layer of uniform depth. (The local Rayleigh number $\overline{Ra}(X)$, is defined in terms of the local fluid depth $h(X)$, where X measures horizontal distance). It was assumed in I that the transition region occurred sufficiently far from the endwalls of the container for their influence to be negligible.

Experimental observations of a similar configuration (uniform depth but monotonically increasing vertical temperature difference) reported by Srulijes (1979) do not include results that may be compared to the theory presented in I because the transition region occurs too close to an endwall. Also, it was not possible in I to determine which was the preferred mode of instability because none possessed a critical Rayleigh number at which it first appeared. Instead, the amplitude of each mode increased smoothly from the shallower, more stable, end to the deeper end. If the local Rayleigh number is increased uniformly, say by increasing the vertical temperature difference, ΔT between the boundaries, then the transition region moves towards the shallower end. The layer of fluid is most unstable adjacent to the endwall where the depth is greatest, which means that if we wish to determine which mode appears first as ΔT increases, we must investigate the solution near this endwall.

The object of this paper is to determine the effect of endwalls on the onset and development of convection in such a layer of fluid. For simplicity, and in order to provide a more exact comparison with experiment, we shall treat a slightly different problem to that discussed in I, in which a horizontal layer of fluid is heated non-uniformly from below.

We suppose that the lengthscale of the horizontal variation in base temperature is of the same order of magnitude as the length L of the container, and that this is much larger than the depth h of the container. Then the aspect ratio $\epsilon = h/L \ll 1$. In this parameter régime the flow may be regarded as consisting of a steady base flow and a disturbed flow. It is shown in §2 that the base flow consists of a temperature distribution which is linear in the vertical coordinate, and a weak circulation. The boundary conditions at the endwalls are satisfied by this solution at leading order, but there remains a discrepancy $O(\epsilon)$ which 'forces' the disturbed flow. To leading order there is no forcing, and the disturbed flow satisfies a problem with 'perfect' end conditions. The appropriate solution is given in §3. The $O(\epsilon)$ 'imperfection' is shown, in §4, to be particularly important for subcritical values of the Rayleigh number when a weak cellular flow occurs near the endwall at the hotter end of the container. Finally, the results are discussed and a comparison with experimental results is made in §5.

2. Formulation of the problem

We consider the onset of cellular convection in a two-dimensional container of fluid, $0 \leqslant x^* \leqslant L, 0 \leqslant z^* \leqslant h$; the container is unbounded in the y^* direction. The temperature T^* of the upper surface $z^* = h$ is kept constant and equal to T_0^*, while that of the lower surface $z^* = 0$ is taken to be $T_0^* + \Delta T_0^* F(x^*/L)$, where $\Delta T_0^* > 0$ and $F(x^*/L)$ is prescribed. In this paper we shall confine our attention to temperature variations for which $F(0) = 1$ and $\mathrm{d}F^*/\mathrm{d}x^* < 0$ for all $x^* \in [0, L]$. This means that the temperature difference between the horizontal boundaries is greatest at $x^* = 0$, where it is equal to ΔT_0^*, and decreases monotonically in $[0, L]$. For simplicity we shall assume that the horizontal boundaries are stress-free, although this is by no means essential; it is not expected that the results will be qualitatively different for rigid boundaries. The endwalls $x^* = 0, L$ are taken to be rigid and, again for simplicity, perfect insulators.

We shall use dimensionless Cartesian coordinates $(x, y, z) = h^{-1}(x^*, y^*, z^*)$, a dimensionless temperature field $T = (\Delta T_0^*)^{-1} T^*$ and dimensionless velocity components (u, v, w), time t and pressure p, scaled on κ/h, h^2/κ, ρ_0/h^2 respectively, where κ is the coefficient of thermal diffusivity and ρ_0 is a reference density.

Under the assumption that the Boussinesq approximation may be made, the dimensionless equations of continuity and conservation of energy and momentum are

$$\nabla \cdot \boldsymbol{u} = 0, \tag{2.1}$$

$$\frac{\partial T}{\partial t} + (\boldsymbol{u} \cdot \nabla) T = \nabla^2 T, \tag{2.2}$$

$$\sigma^{-1}\left(\frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u}\right) = \frac{1}{\sigma}\nabla_p + Ra\, T\hat{\boldsymbol{z}} + \nabla^2 \boldsymbol{u}. \tag{2.3}$$

The Rayleigh number Ra and Prandtl number σ are defined by

$$Ra = \frac{\alpha g (\Delta T_0^*) h^3}{\nu\kappa}, \quad \sigma = \frac{\nu}{\kappa}, \tag{2.4}$$

where α, ν are the coefficients of cubical expansion and kinematic viscosity respectively and g is the acceleration due to gravity.

In dimensionless terms the boundary conditions are

$$\left.\begin{aligned} &w = \frac{\partial u}{\partial z} = \frac{\partial v}{\partial z} = 0 \quad (z = 0, 1), \\ &T = 0 \quad (z = 1), \quad T = F(\epsilon x) \quad (z = 0), \end{aligned}\right\} \tag{2.5}$$

$$u = v = w = \frac{\partial T}{\partial x} = 0 \quad (x = 0, \epsilon^{-1}). \tag{2.6}$$

Here $\epsilon = h/L$ is the aspect ratio of the container. We shall assume that $\epsilon \ll 1$ and that F is a function only of the 'slow' horizontal variable $X = \epsilon x$ scaled on the length L of the container.

The problem defined by (2.1)–(2.6) is analogous to that considered in I except for the presence of the endwalls at $X = 0, 1$.

The effect of endwalls on the onset of convection in a uniformly heated rectangular container has been considered by Hall & Walton (1977) and Daniels (1977). They distinguished two classes of boundary conditions at the endwalls: in the first ('perfect') case the conditions at the endwalls are satisfied exactly by the steady base solution which holds prior to the onset of Rayleigh–Bénard convection in an unbounded layer, while in the second case (the 'imperfect' case) these conditions are not satisfied by the base solution. Hall & Walton and Daniels were able to obtain solutions for the 'perfect' and 'slightly (or weakly) imperfect' case by perturbing about the base solution.

The same ideas may be carried over to the present problem. Let us denote by subscript B the steady base solution which holds for the unbounded layer. For $\epsilon \ll 1$ this solution may be found by expanding in powers of ϵ as in I. We find that

$$T_{\mathrm{B}} = T_0 + O(\epsilon), \quad u_{\mathrm{B}} = \epsilon u_0 + O(\epsilon^2) \quad v_{\mathrm{B}} \equiv 0, \quad w_{\mathrm{B}} = \epsilon^2 w_0 + O(\epsilon^2),$$

where

$$\left.\begin{aligned} &T_0 = (1-z)\,F(X), \\ &u_0 = RaF'(X)\frac{\mathrm{d}G}{\mathrm{d}z}, \quad w_0 = -RaF''(X)\,G(z), \end{aligned}\right\} \tag{2.7}$$

with

$$G(z) = \tfrac{1}{360}(-3z^5 + 15z^4 - 20z^3 + 8z).$$

To leading order the base state consists of a temperature distribution whose vertical gradient $F(X)$ decreases as X increases, and a weak shear flow. It is useful later on to use a 'local' Rayleigh number $\overline{Ra}$ defined in terms of the local vertical temperature gradient $F(X)$ and related to Ra by

$$\overline{Ra} = F(X)\,Ra. \tag{2.8}$$

We shall show below that the amplitude of the perturbed solution is at least $O(\epsilon)$ in the region of main interest, and there is therefore no point in retaining any terms greater than the $O(\epsilon)$ terms in the base solution, the higher-order terms being absorbed in the perturbation. Let us now write

$$T = T_0 + \theta, \quad \boldsymbol{u} = \boldsymbol{u} + \epsilon(u_0, 0, 0), \tag{2.9}$$

where $(\theta, \boldsymbol{u})$ are the perturbed variables with $|\theta| \ll |T_0|$, $|\boldsymbol{u}| \ll 1$.

The boundary conditions on the perturbed quantities are

$$\theta = w = \frac{\partial u}{\partial z} = \frac{\partial v}{\partial z} = 0 \quad (z = 0, 1), \tag{2.10a}$$

$$u = -\epsilon u_0, \quad v = w = 0 \quad (X = 0, 1), \tag{2.10b}$$

$$\frac{\partial \theta}{\partial x} = -\epsilon(1-z)\, F'(0) \quad (X = 0), \tag{2.10c}$$

$$= -\epsilon(1-z)\, F'(1) \quad (X = 1). \tag{2.10d}$$

The horizontal variation of the base state is then manifest as a weak forcing at the endwalls, and the boundary conditions (2.10) may be described as 'slightly imperfect' in the terminology of Daniels and Hall & Walton. To leading order in ϵ it is possible to neglect the forcing altogether and consider the 'perfect' boundary conditions in which (2.10b–d) are replaced by

$$\frac{\partial \theta}{\partial x} = u = v = w = 0 \quad (X = 0, 1). \tag{2.11}$$

We remind the reader that in both cases the horizontal variation of the base state also enters the perturbation problem through nonlinear terms in the governing equations.

We shall return to the imperfect problem in §4, but in §3 we discuss the perfect problem.

3. The 'perfect' problem

The equations satisfied by the perturbed variables are given in I, but of course we must now take into account the boundary conditions at $X = 0, 1$. Solutions were obtained in I in the region where the local Rayleigh number is either below critical or is, in some sense, close to it. In the present problem the local Rayleigh number $\overline{Ra}$ is greatest at $X = 0$ where it is equal to Ra, so we shall impose the restriction that Ra is either below critical or, if it exceeds critical, it remains close to it. The critical value referred to here may be taken to be the critical value for the onset of convection in a uniformly heated unbounded layer. For isothermal stress-free horizontal boundaries this value, denoted by Ra_c^∞, is equal to $\frac{27}{4}\pi^4$.

Let us write

$$Ra = Ra_c^\infty(1 + Ra_1). \tag{3.1}$$

Then either $Ra_1 < 0$ or, if $Ra_1 > 0$ we must assume that $Ra_1 \ll 1$. Particular scalings for Ra_1 in terms of ϵ will be chosen later.

Under these conditions we look for solutions of the perturbed problem in the form

$$\begin{pmatrix} u \\ v \\ w \\ \theta \end{pmatrix} = \Delta A(X_1) E \begin{pmatrix} u_1 \\ v_1 \\ w_1 \\ \theta_1 \end{pmatrix} + O(\Delta^2), \tag{3.2}$$

where $E = \exp\{\mathrm{i}k_x x + \mathrm{i}k_y y\}$ with $k_x^2 + k_y^2 = k_0^2 = \frac{1}{2}\pi^2$. This solution represents a slow variation with amplitude $\Delta A(X_1)$ about the critical solution for the unbounded uniformly heated layer. The amplitude function $A(X_1)$, the small parameter Δ, and the slow variable X_1 depend upon ϵ and k_x, the x-component of the wavenumber, as described in I. There it is shown that if $k_x \approx 1$, then $\Delta = \epsilon^{\frac{1}{3}}$, $X_1 = \epsilon^{-\frac{2}{3}} X = \epsilon^{\frac{1}{3}} x$, and

A satisfies

$$(F'(0)\,X_1+Ra_1\epsilon^{-\frac{2}{3}})\,A-\alpha_1 A\,|A|^2+4\alpha_2\frac{\mathrm{d}^2A}{\mathrm{d}X_1^2}=0, \tag{3.3}$$

an equation known as the second Painlevé transcendent. For stress-free isothermal boundary conditions at $z = 0, 1$, the constants α_1, α_2 are given by

$$\alpha_1 = (3\pi^2)^{-1}, \quad \alpha_2 = \tfrac{4}{3}\pi^{-4}k_x^2 \tag{3.4}$$

When k_x is small a different scaling must be used because $\alpha_2 \to 0$ as $k_x \to 0$. It is shown in I that for $k_x = \epsilon^{\frac{2}{5}}\bar{k}_x$ with $\bar{k}_x \approx 1$ we need to take $\Delta = \epsilon^{\frac{2}{5}}$, $X_1 = \epsilon^{-\frac{4}{5}}X = \epsilon^{\frac{1}{5}}x$, in which case A satisfies

$$(F'(0)\,X_1+Ra_1\epsilon^{-\frac{4}{5}})\,A-\alpha_1 A\,|A|^2-\alpha_2\left(2\mathrm{i}\bar{k}_x\frac{\mathrm{d}}{\mathrm{d}X_1}+\frac{\mathrm{d}^2}{\mathrm{d}X_1^2}\right)^2 A = 0. \tag{3.5}$$

In order to simplify the discussion we shall now confine our attention to the two extreme roll configurations: transverse rolls (rolls aligned with their axes in the y-direction) for which $k_x = k_0 = \sqrt{\tfrac{1}{2}}\pi$, $k_y = 0$, and longitudinal rolls (rolls aligned with their axes in the x-direction) for which $k_y = k_0 = \sqrt{\tfrac{1}{2}}\pi$, $k_x = 0$. For transverse rolls $\Delta = \epsilon^{\frac{1}{3}}$, $X_1 = \epsilon^{\frac{1}{3}}x$, and A satisfies (3.3) with $\alpha_2 = 2/3\pi^2$ and for longitudinal rolls $\Delta = \epsilon^{\frac{2}{5}}$, $X_1 = \epsilon^{\frac{1}{5}}x$, and A satisfies

$$(F'(0)\,X_1+Ra_1\epsilon^{-\frac{4}{5}})\,A-\alpha_1 A|A|^2-\alpha_2\frac{\mathrm{d}^4A}{\mathrm{d}X_1^4}=0 \tag{3.6}$$

with $\alpha_2 = \tfrac{4}{3}\pi^4$.

Using either scaling for X_1, the boundary at $X = 0$ becomes $X_1 = 0$, while the boundary $X = 1$ becomes $X_1 = \infty$. The amplitude of the cellular disturbance is modulated on a lengthscale long compared with the depth of the fluid layer and the wavelength of the disturbance but short compared with the length of the container. The boundary conditions (2.11) become

$$A = 0 \quad (X_1 = 0), \quad A \to 0 \quad (X_1 \to \infty) \tag{3.7}$$

for the transverse mode, and

$$A = \frac{\mathrm{d}A}{\mathrm{d}X_1} = 0 \quad (X = 0), \quad A = \frac{\mathrm{d}A}{\mathrm{d}X_1} \to 0 \quad (X_1 \to \infty) \tag{3.8}$$

for the longitudinal mode.

The amplitude equations (3.3), (3.6) together with their boundary conditions (3.7), (3.8) must be solved numerically, but first we eliminate the coefficients in (3.3), (3.6) by reducing the equations to canonical form.

In (3.3) we write

$$\left.\begin{aligned} Ra_1 &= \epsilon^{\frac{2}{3}}(-2\alpha_2{}^{\frac{1}{2}}F'(0))^{\frac{2}{3}}\,Ra_{\mathrm{T}}, \\ X_1 &= \left(\frac{-4\alpha_2}{F'(0)}\right)^{\frac{1}{3}} X_{\mathrm{T}}, \quad A = (-2\alpha_2{}^{\frac{1}{2}}F'(0))^{\frac{1}{3}}\left(\frac{2}{\alpha_1}\right)^{\frac{1}{2}} A_{\mathrm{T}}, \end{aligned}\right\} \tag{3.9}$$

whereupon A_{T} satisfies

$$\frac{\mathrm{d}^2A_{\mathrm{T}}}{\mathrm{d}X_{\mathrm{T}}^2}+(Ra_{\mathrm{T}}-X_{\mathrm{T}})\,A_{\mathrm{T}}-2A_{\mathrm{T}}^3 = 0, \tag{3.10}$$

with

$$A_{\mathrm{T}} = 0 \quad (X_{\mathrm{T}} = 0), \quad A_{\mathrm{T}} \to 0 \quad (X_{\mathrm{T}} \to \infty). \tag{3.11}$$

We remind the reader that $F'(0) < 0$. We shall assume that $Ra_{\mathrm{T}} \sim 1$ or, equivalently, $Ra_1 \sim \epsilon^{\frac{2}{3}}$.

Numerical solutions of the problem posed by (3.10), (3.11) have been obtained in two ways. First, following Miles (1978) we may write

$$A_{\mathrm{T}} \sim \alpha \mathrm{Ai}(X_{\mathrm{T}}) \quad (X_{\mathrm{T}} \to \infty),$$

where $\mathrm{Ai}(X_{\mathrm{T}})$ is an Airy function and α is a constant, and integrate (3.10) inwards from a suitably large value of X_{T}. The constant Ra_{T} is chosen so that, for a prescribed value of α_{T}, A_{T} vanishes at $X_{\mathrm{T}} = 0$. Miles (1978) and Rosales (1978) report that it is very difficult to obtain accurate solutions for even moderately large values of Ra_{T} and, indeed, Miles presents results only for $Ra_{\mathrm{T}} \lesssim 7$.

An alternative method, which presents none of these difficulties, involves expansions in Chebychev polynomials and is described in detail in I. The results obtained by these two methods are in good agreement for small values of Ra_{T} and are summarized in figures 1 and 2. There are several noteworthy features of the results. First, the minimum value of Ra_{T} for which solutions exist is

$$Ra_{\mathrm{T}} = Ra_{\mathrm{T,\,C}} = 2.338$$

and the critical value of Ra for the onset of transverse rolls is therefore

$$Ra_{\mathrm{c}} = \frac{27\pi^4}{4}(1 + 2.338(2\alpha_2^{\frac{1}{2}} F'(0))^{\frac{2}{3}} \epsilon^{\frac{2}{3}} + \ldots).$$

The critical value of the Rayleigh number for the onset of the transverse mode therefore exceeds that for unmodulated, unbounded Bénard convection by a term $O(\epsilon^{\frac{2}{3}})$. This contrasts with an increment $O(\epsilon^2)$ found by Hall & Walton for the onset of Bénard convection in a rectangular box with 'perfect' end conditions. Critical values of Rayleigh numbers are usually determined on the basis of linearized theory in which the amplitude of the disturbance is assumed to be infinitesimally small. The present case is no exception, because the critical value occurs in the limit $|A| \to 0$ when $A_{\mathrm{T}} \propto \mathrm{Ai}(X_{\mathrm{T}})$ for all $X_{\mathrm{T}} \geqslant 0$ and $-Ra_{\mathrm{T}}$ is equal to the first zero of Ai.

Secondly, the location of the maximum value of the amplitude, denoted by X_{TM}, appears to approach the wall $X_{\mathrm{T}} = 0$ as $Ra_{\mathrm{T}} \to \infty$, and in fact Miles (1980) has demonstrated that $X_{\mathrm{TM}} \sim Ra_{\mathrm{T}}^{-\frac{1}{2}} \ln R_{\mathrm{T}}$ as $R_{\mathrm{T}} \to \infty$. Thirdly, the maximum value of the amplitude A_{TM} rapidly approaches the value $[\frac{1}{2}(Ra_{\mathrm{T}} - X_{\mathrm{TM}})]^{\frac{1}{2}}$ as $Ra_{\mathrm{T}} \to \infty$, which is the value given by quasistationary weakly nonlinear theory in which X-derivatives are ignored). This limiting behaviour means that a boundary-layer structure near $X_{\mathrm{T}} = 0$ becomes more pronounced as $Ra_{\mathrm{T}} \to \infty$. Miles has shown that the solution in the boundary layer takes the form

$$A_{\mathrm{T}} = \tfrac{1}{2}\delta^{-\frac{1}{3}}(A_{\mathrm{T0}} + \delta A_{\mathrm{T1}} + \ldots),$$

where
$$\delta = [2(Ra_{\mathrm{T}} - X_{\mathrm{TM}})]^{-\frac{3}{2}} \ll 1,$$
$$\mathrm{A}_{\mathrm{T0}} = \tanh \tfrac{1}{2}\theta$$
$$A_{\mathrm{T1}} = -\theta \tanh \tfrac{1}{2}\theta + \tfrac{1}{8}[2\mathrm{e}^{-\theta} - (\theta^2 + 3\theta + 5 - 3\mathrm{e}^{-\theta}) \operatorname{sech}^2 \tfrac{1}{2}\theta],$$
$$\theta = \delta^{-\frac{1}{3}} X_{\mathrm{T}}.$$

As Ra_1 increases beyond $O(\epsilon^{\frac{2}{3}})$, the present theory breaks down because the maximum value of the amplitude exceeds that which can be accommodated by weakly nonlinear theory. All that that theory can then describe is the solution in a small neighbourhood of the value of X_{T} where the local Rayleigh number is equal to the critical value for unmodulated unbounded Bénard convection. This analysis has already been given in I.

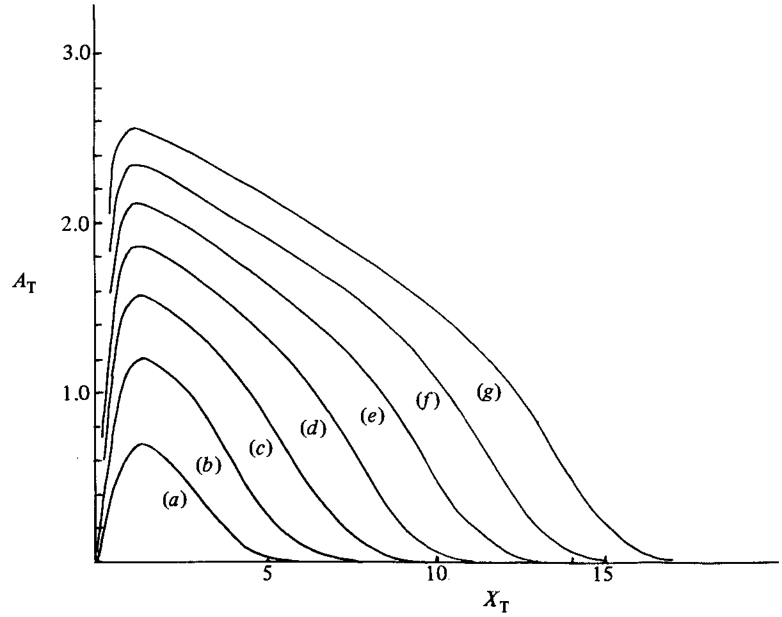

FIGURE 1. A_T as a function of X_T for values of Ra_T as follows: (*a*) 3.091; (*b*) 4.727; (*c*) 6.585; (*d*) 8.504; (*e*) 10.450; (*f*) 12.410; (*g*) 14.380; corresponding to $A'_T(0) = 1, 2, 3, \ldots, 7$. Note that $Ra_{TC} = 2.338$.

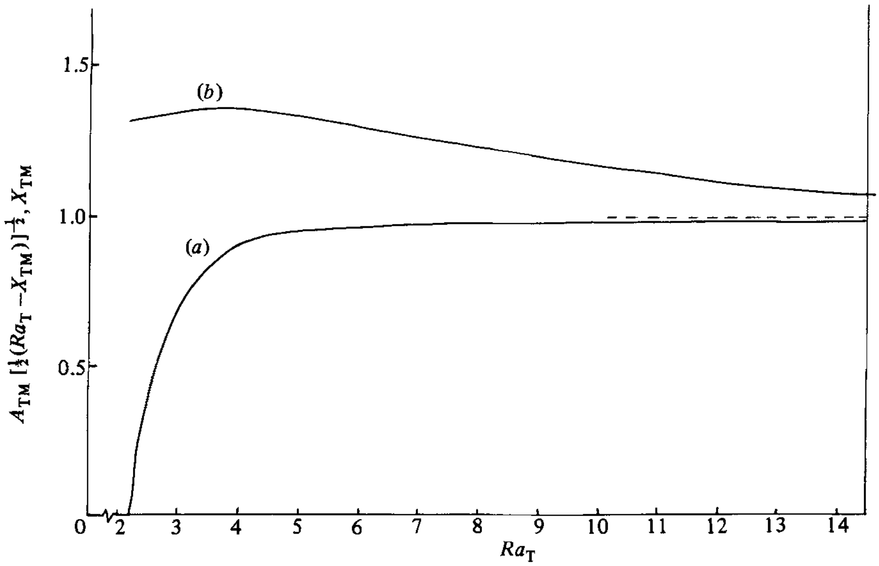

FIGURE 2. (*a*) $A_{TM}[\frac{1}{2}(\mathrm{Ra}_T - X_{TM})]^{-\frac{1}{2}}$ and (*b*) X_{TM} as functions of Ra_T for $Ra_T \geqslant Ra_{TC} = 2.338$.

The amplitude equation (3.6) for the longitudinal mode may be reduced to canonical form by the transformation

$$\left.\begin{aligned} X_1 &= \left(\frac{-\alpha_2}{F'(0)}\right)^{\frac{1}{5}} X_L, \quad Ra_1 = \epsilon^{\frac{4}{5}}(-\alpha_2^{\frac{1}{4}}F'(0))^{\frac{4}{5}}, \\ A_1 &= (-\alpha_2^{\frac{1}{4}}F'(0))^{\frac{2}{5}}\left(\frac{2}{\alpha_1}\right)^{\frac{1}{2}} A_L, \end{aligned}\right\} \qquad (3.13)$$

whereupon A_{L} satisfies

$$\frac{\mathrm{d}^4 A_{\mathrm{L}}}{\mathrm{d}X_{\mathrm{L}}^4} - (Ra_{\mathrm{L}} - X_{\mathrm{L}})\, A_{\mathrm{L}} + 2A_{\mathrm{L}}^3 = 0, \tag{3.14}$$

with

$$\left.\begin{aligned} A_{\mathrm{L}} = \frac{\mathrm{d}A_{\mathrm{L}}}{\mathrm{d}X_{\mathrm{L}}} = 0 \quad (X_{\mathrm{L}} = 0), \\ A_{\mathrm{L}}, \frac{\mathrm{d}A_{\mathrm{L}}}{\mathrm{d}X_{\mathrm{L}}} \to 0 \quad (X_{\mathrm{L}} \to \infty). \end{aligned}\right\} \tag{3.15}$$

Very little seems to be known about solutions of (3.14) except for some results (for different boundary conditions) given in I and details of the solutions of the linearised equation given by Ross (1966). Numerical solutions may be found, at least in principle, by the methods adopted for solving (3.10), (3.11). For $X_{\mathrm{L}} \to \infty$ the solution which decays to zero has the asymptotic behaviour

$$A_{\mathrm{L}} \sim \beta_1 X_{\mathrm{L}}^{-\frac{3}{8}} \exp\{-\sqrt{\tfrac{1}{2}}\eta\} \cos(\sqrt{\tfrac{1}{2}}\eta + \beta_2), \tag{3.16}$$

where β_1, β_2 are arbitrary constants and $\eta = \frac{4}{5} X_{\mathrm{L}}^{\frac{5}{4}}$ (Ross 1966). We have derived a more comprehensive expansion, details of which will not be given here but are available from the author. Solutions of (3.14) may then be found by integrating inwards from a suitably large value of X_{L}, and the parameters β_1, β_2 varied until, for a given value of Ra_{L}, $A_{\mathrm{L}} = \mathrm{d}A_{\mathrm{L}}/\mathrm{d}X_{\mathrm{L}} = 0$ at $X_{\mathrm{L}} = 0$. This procedure is even more difficult to use than for the second-order problem because of the presence of solutions that grow exponentially as $X_{\mathrm{L}} - Ra_{\mathrm{L}} \to -\infty$. A better method is to integrate inwards from a large value of X_{L} and outwards from $X_{\mathrm{L}} = 0$ and match in between, but even this method is fraught with difficulties for Ra_{L} even moderately large. Satisfactory results were obtained for Ra_{L} less than about 10.

More extensive results have been obtained by the second method referred to above, and good agreement between the two sets of results was obtained for $Ra_{\mathrm{L}} \lesssim 10$. Sample curves of A_{L} against X_{L} are plotted in figure 3, and are seen to be similar to those for the transverse mode except for the double zero at $X_{\mathrm{L}} = 0$ and the weak oscillatory tail as $X_{\mathrm{L}} \to \infty$. Values of the maximum amplitude A_{LM} and its location X_{LM}, are plotted as functions of Ra_{L} in figure 4.

As for the transverse mode, there is a minimum value Ra_{LC} of Ra_{L} for which solutions exist, and again it is associated with the first zero of the linearized solution. We find that

$$Ra_{\mathrm{LC}} = 3.094, \tag{3.17}$$

and it follows that the minimum value of Ra for longitudinal rolls to exist is

$$Ra = Ra_{\mathrm{c}} = \tfrac{27}{4}\pi^4 (1 + 3.094(-\alpha_2^{\frac{1}{4}} F'(0))^{\frac{4}{5}} \epsilon^{\frac{4}{5}} + \ldots). \tag{3.18}$$

The increment of the critical Rayleigh number above that for the onset of convection in an unbounded unmodulated layer is $O(\epsilon^{\frac{4}{5}})$, and contrasts with an increment $O(\epsilon^{\frac{2}{3}})$ for the transverse mode. The implications of this result for the preferred mode will be discussed in §5.

It appears from figure 4 that $X_{\mathrm{LM}} \to 0$, $A_{\mathrm{LM}} \to (\frac{1}{2}|Ra_{\mathrm{LM}} - X_{\mathrm{LM}}|)^{\frac{1}{2}}$ as $Ra_{\mathrm{L}} \to \infty$. An asymptotic analysis similar to that given by Miles for (3.10) may be undertaken in this limit, and details are given in the appendix. The main result is

$$X_{\mathrm{LM}} \sim Ra_{\mathrm{L}}^{-\frac{1}{4}} \ln Ra_{\mathrm{L}} \quad (Ra_{\mathrm{L}} \to \infty),$$

which confirms that the point of maximum amplitude moves closer to the wall, but it does so even slower than for the transverse mode.

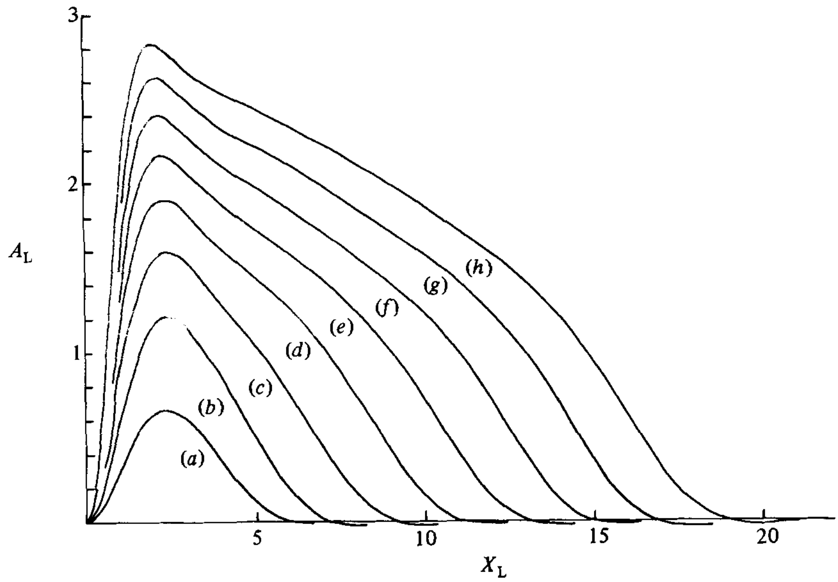

FIGURE 3. A_L as a function of X_L for values of Ra_L as follows: (*a*) 3.745; (*b*) 5.368; (*c*) 7.254; (*d*) 9.167; (*e*) 11.103; (*f*) 13.054; (*g*) 15.014; (*h*) 16.980; corresponding to $A''_L(0) = 1, 2, 3, \ldots, 8$. Note that $Ra_{LC} = 3.094$.

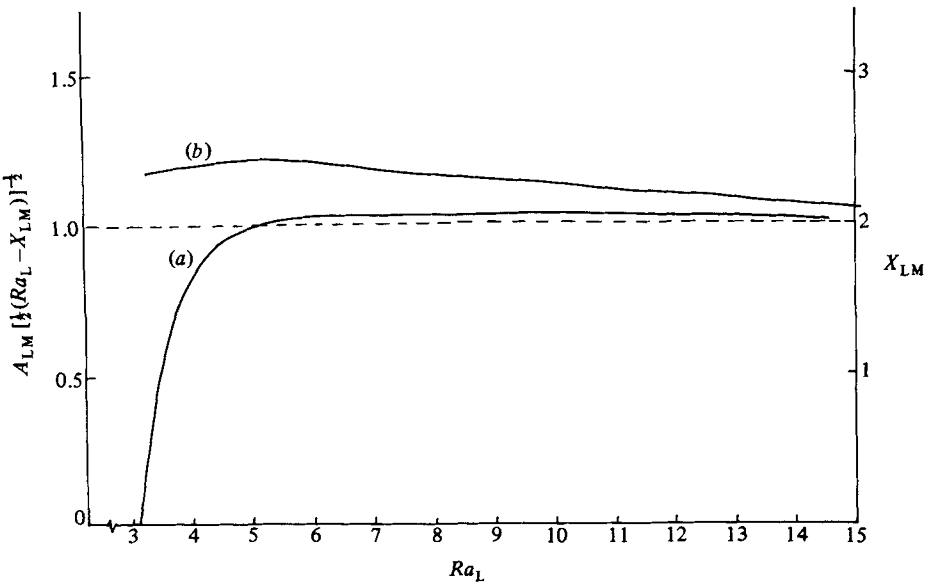

FIGURE 4. (*a*) $A_{LM}[\frac{1}{2}(Ra_L - X_{LM})]^{-\frac{1}{2}}$ and (*b*) X_{LM} as functions for R_L for $Ra_L \geqslant Ra_{LC} = 3.094$.

4. The 'imperfect' problem

The solution for the perturbed variables given in §3 satisfies the conditions of no-slip and zero heat transfer at the endwalls (at least to leading order in ϵ). The steady base flow does not, however, satisfy these conditions, and this discrepancy results in a weak forcing of the perturbed solution at the endwalls. In this section we investigate the effect of this weak forcing on the onset of convection.

The situation now presented is similar to that discussed by Hall & Walton (1977).

They considered the onset of convection in a rectangular two-dimensional box heated uniformly from below when the endwalls were not quite perfect insulators, resulting in a weak forcing of the cellular mode. For values of the Rayleigh numbers which were not too large the forcing manifested itself as a weak disturbance concentrated near the endwalls, but as the Rayleigh number increased towards critical the disturbance penetrated into the interior of the box and its amplitude increased dramatically, though smoothly, and approached that given by conventional weakly nonlinear theory. The most interesting features of the solution are that a flow exists for values of the Rayleigh number below the critical value for the onset of convection in the corresponding 'perfect' problem and the onset of the convective instability is quite smooth – there is no longer a bifurcation at a critical Rayleigh number. We expect a similar description to apply to the present problem, though the details of the calculation are different.

The method of solution is similar to that for the unmodulated problem. We look for a solution for the perturbed variables which satisfies the boundary conditions (2.10a) at $z = 0, 1$ by expanding in Fourier series. At the same time we observe that the forcing at the endwalls in (2.10b–d) is independent of y, and this suggests that we seek forced solutions independent of y. Let us write

$$\begin{pmatrix} u \\ v \\ w \\ \theta \end{pmatrix} = \sum_{n=1}^{\infty} A_n(X_1) \begin{pmatrix} u_n(x)\cos n\pi z \\ 0 \\ w_n(x)\sin n\pi z \\ \theta_n(x)\sin n\pi z \end{pmatrix} + \text{higher-order terms}, \tag{4.1}$$

where $A_n(X_1)$ is the slowly varying amplitude of the nth Fourier mode; both $A_n(X_1)$ and X_1 will be related to ϵ in due course. Solutions in this form are valid only close to the endwalls where $F(X)$ may be expanded about $X = 0$ or $X = 1$ as appropriate, and we shall see that in contrast to the uniformly heated problem, the solution remains confined to the neighbourhood of the endwalls even for Ra just exceeding critical. The solutions near the two endwalls may then be treated entirely separately. The functions $u_n(x)$, $w_n(x)$, $\theta_n(x)$ in (4.1) satisfy the usual linearized equations for Rayleigh–Bénard convection.

We also need to expand the boundary conditions (2.10b–d) in Fourier series on [0, 1], and this requires the Fourier decomposition of $1-z$ and $\mathrm{d}G/\mathrm{d}z$, defined in (2.7). We write

$$1-z = \sum_{n=1}^{\infty} g_n \sin n\pi z, \quad \frac{\mathrm{d}G}{\mathrm{d}z} = \sum_{n=1}^{\infty} h_n \cos n\pi z, \tag{4.2}$$

where, in particular, $g_1 = 2/\pi, h_1 = 2/\pi^4$; the higher coefficients are not needed explicitly in the subsequent analysis. The boundary conditions on the nth Fourier mode are then

$$A_n(0)\frac{\mathrm{d}\theta_n}{\mathrm{d}x} = -\epsilon F'(0)\, g_n, \quad A_n(0)\, w_n = 0, \quad A_n(0)\, u_n = -\epsilon Ra F'(0)\, h_n \tag{4.3}$$

at $X = x = 0$. Similar conditions apply at the colder end of the container at $X = 1$.

Provided that Ra is not too large (in a sense to be defined more precisely later) the forcing generates a disturbance which is concentrated in regions of thickness $O(\epsilon)$ on the X-scale at the endwalls. In this regime the slow variation may be neglected altogether to a first approximation, and we set $A_n(X_1) \approx A_n(0)$, for the solution near $X = 0$.

For that solution we follow Hall & Walton and write

$$\theta_n = \sum_{j=1}^{3} a_{nj} \exp\{\mathrm{i}k_{nj}x\}, \quad w_n = \sum_{j=1}^{\infty} b_{nj} \exp\{\mathrm{i}k_{nj}x\},$$
$$u_n = \mathrm{i}\pi \sum_{j=1}^{3} b_{nj} k_{nj}^{-1} \exp\{\mathrm{i}k_{nj}x\},$$

where $b_{nj} = -k_{nj} a_{nj} (k_{nj}^2 + n^2\pi^2)^{-2}$, and $k_{nj}, j = 1, 2, 3$, are the three roots of

$$(k_n^2 + \pi^2 n^2)^3 \, k_n^2 \, Ra \tag{4.5}$$

with positive real part. The unknown coefficients $a_{nj}, j = 1, 2, 3$, are determined by the boundary conditions (4.3). No details of the calculation will be given here, but it is clear that the amplitude of the disturbance is $O(\epsilon Ra F'(0))$ and that for $Ra < Ra_c^\infty$ all Fourier modes decay exponentially as $x \to \infty$. Near the colder wall $RaF(1)$ replaces Ra in the corresponding calculations, and this means that, since $F(1) < 1$, the solution there decays more rapidly than it does near the hotter wall. Even when Ra approaches Ra_c^∞, $RaF(1)$ is still well below critical, and this description of the flow near the colder wall is adequate for all values of Ra considered here.

As Ra increases towards Ra_c^∞ the description of the flow near $X = 0$ must be modified. When $Ra = Ra_c^\infty$ two solutions of (4.5) with $n = 1$ merge and become real; the solution then fails to decay with x, the disturbance is not so closely confined to the immediate neighbourhood of the endwall, and we can no longer neglect the variation in the local Rayleigh number with X. It is clear that we need to examine the solution near $Ra = Ra_c^\infty$ more closely. The scalings adopted in §3 for the transverse mode suggest that we should examine the parameter regime

$$\left.\begin{aligned} Ra &= Ra_c^\infty + \epsilon^{\frac{2}{3}} (-2\alpha_2^{\frac{1}{2}} F'(0))^{\frac{2}{3}} Ra_{\mathrm{T}}, \\ X &= \epsilon^{\frac{2}{3}} \left(\frac{-4\alpha_2}{F'(0)}\right)^{\frac{1}{3}} X_{\mathrm{T}}, \end{aligned}\right\} \tag{4.6}$$

and that we look for solutions in the form

$$\theta = A_1(X_{\mathrm{T}}) \sin \pi z \, \mathrm{e}^{\mathrm{i}k_c x} + B \mathrm{e}^{-2\pi x} \sin \pi z + \mathrm{c.c.} + \sum_{n=2}^{\infty} A_n(X_{\mathrm{T}}) \, \theta_n(x) \sin n\pi z, \tag{4.7}$$

where B is an arbitrary constant corresponding to a_{13} above. The slowly modulated amplitude A_1 then satisfies

$$\frac{\mathrm{d}^2 A_1}{\mathrm{d}X_{\mathrm{T}}^2} + (Ra_{\mathrm{T}} - X_{\mathrm{T}}) A_1 - \alpha_1 (-2\alpha_2^{\frac{1}{2}} F'(0))^{-\frac{2}{3}} \epsilon^{-\frac{2}{3}} A_1 |A_1|^2 = 0, \tag{4.8}$$

corresponding to (3.3). All the terms in (4.7) decay exponentially as $x \to \infty$ except the first, to which we now restrict our attention. The boundary conditions for A_1 may now be obtained in the same way as in Daniels (1977) for the unmodulated problem. They take the form

$$A_1 = \epsilon F'(0) \, C \mathrm{e}^{\mathrm{i}\gamma} \quad (X_{\mathrm{T}} = 0), \quad A_1 \to 0 \quad (X_{\mathrm{T}} \to \infty), \tag{4.9}$$

where C, γ are constants depending on g_1, h_1. The details of the forms of C, γ are of no particular interest and are omitted for the sale of brevity. Since the coefficients in (4.8) are real and A_1 vanishes as $X_1 \to \infty$, the phase of A_1 is independent of X_{T}. (In contrast with the unmodulated case, when the forcing has different phase at each end of the box.) We may therefore look for solutions with A_1 replaced by $|A_1|$ in (4.8) subject to the boundary conditions

$$|A_1| = C\epsilon F'(0) \quad (X_{\mathrm{T}} = 0), \quad |A_1| \to 0 \quad (X_{\mathrm{T}} \to \infty). \tag{4.10}$$

This suggests that we look for solutions of (4.8) in which $|A_1| \sim \epsilon$, in which case the nonlinear term may be neglected. The appropriate solution is

$$|A_1| = C\epsilon F'(0)\frac{\mathrm{Ai}(X_\mathrm{T}-Ra_\mathrm{T})}{\mathrm{Ai}(-Ra_\mathrm{T})}, \tag{4.11}$$

where $\mathrm{Ai}(X)$ is an Airy function. This solution remains valid provided that $\mathrm{Ai}(-Ra_\mathrm{T})$ is not close to zero. It becomes zero when $Ra_\mathrm{T} = Ra_\mathrm{TC} = 2.338$ which is the critical value of Ra_T for the onset of the transverse mode in the absence of forcing at the endwalls. The nature of the singularity may be investigated by writing

$$Ra_\mathrm{T} = Ra_\mathrm{TC}+\mu, \quad \mu \ll 1. \tag{4.12}$$

It follows from (4.11) that $|A_1| \sim \epsilon\mu^{-1}$ as $\mu \to 0$. The singularity in $|A_1|$ may be removed by retaining the nonlinear terms in (4.8). When $|A_1| \sim \epsilon\mu^{-1}$ the nonlinear term comes into play when $\epsilon^2\mu^{-2}\epsilon^{-\frac{2}{3}} \sim \mu$, i.e. $\mu \sim \epsilon^{\frac{4}{9}}$ and the amplitude is then $O(\epsilon^{\frac{5}{9}})$. The solution in this regime may be obtained as follows. Let us write

$$Ra_\mathrm{T} = Ra_\mathrm{TC}+\bar{\mu}\epsilon^{\frac{4}{9}}, \quad A_1 = \epsilon^{\frac{5}{9}}\bar{A}_1. \tag{4.13}$$

Then we have

$$\frac{\mathrm{d}^2|\bar{A}_1|}{\mathrm{d}X_\mathrm{T}^2}+(Ra_\mathrm{TC}-X_\mathrm{T})|\bar{A}_1|+-\alpha_1(-2\alpha_2^{\frac{1}{2}}F'(0))^{\frac{2}{3}}\epsilon^{\frac{4}{9}}|\bar{A}_1|^3 = 0. \tag{4.14}$$

The appropriate solution is (cf. Miles 1978)

$$|\bar{A}_1| = \alpha\mathrm{Ai}(X_\mathrm{T}-Ra_\mathrm{T})+\alpha^3\epsilon^{\frac{4}{9}}B(X_\mathrm{T}-Ra_\mathrm{TC})+\dots, \tag{4.15}$$

where $B(X_\mathrm{T}-Ra_\mathrm{TC})$ depends upon integrals of the fourth power of Airy functions. The boundary conditions at $X_\mathrm{T} = 0$ is satisfied by choosing α such that

$$-\alpha\mathrm{Ai}'(-Ra_\mathrm{TC})\bar{\mu}+\alpha^3B(-Ra_\mathrm{TC}) = CF'(0). \tag{4.16}$$

Note that retention of the term in α^3 means that α remains finite as $\bar{\mu} \to 0$. To leading order in ϵ the maximum value $\bar{A}_{1\mathrm{M}}$ of the amplitude occurs where $X_\mathrm{T} = R_\mathrm{AT}+a_\mathrm{m}$, where a_m is determined by $\mathrm{Ai}'(a_\mathrm{m}) = 0$, and $\bar{A}_{1\mathrm{M}} = 0.5357\alpha$, with α given by (4.16). A schematic graph of $\bar{A}_{1\mathrm{M}}$ as a function of $\bar{\mu}$ is shown in figure 5. Note that $\alpha \sim \bar{\mu}^{-1}$ as $\bar{\mu} \to -\infty$, which means that $A_1 \sim \epsilon(Ra_\mathrm{T}-Ra_\mathrm{TC})^{-1}$ as $Ra_\mathrm{T} \to Ra_\mathrm{TC}+$ as is required to match with the solution given by (4.11). As $\bar{\mu} \to +\infty$ we find that $\alpha^2 \sim \bar{\mu}$, which means that $A_1 \sim \epsilon^{\frac{1}{3}}(Ra_\mathrm{T}-Ra_\mathrm{TC})^{\frac{1}{2}}$. The forcing by the boundary may be disregarded in this limit, and A_1 matches onto a solution of the unforced problem given by (3.8), (3.10). The detailed matching of the solutions (4.11), for $Ra_\mathrm{T} \ll Ra_\mathrm{TC}$, (4.15) for $Ra_\mathrm{T}-Ra_\mathrm{TC} \sim \epsilon^{\frac{4}{9}}$ and that for $Ra_\mathrm{T}-Ra_\mathrm{TC} \sim 1$ is straightforward and no details will be given here.

The development of the forced flow as Ra increases may be summarized as follows. For $Ra \ll Ra_\mathrm{c}^\infty$ the disturbance is weak, $O(\epsilon)$, and confined to narrow regions of thickness ϵ at the endwalls. Many horizontal and vertical scales are present, and no structure is discernible except that the amplitude is greatest at the wall $X = 0$. When $Ra = Ra_\mathrm{c}^\infty+\epsilon^{\frac{2}{3}}Ra_\mathrm{T}$ with $Ra_\mathrm{T} \ll Ra_\mathrm{TC}$ the solution near $X = 0$ is dominated by the first Fourier mode whose horizontal wavelength is equal to that for unmodulated unbounded Bénard convection. Its amplitude is still $O(\epsilon)$, but it is now modulated on a horizontal lengthscale $O(\epsilon^{\frac{2}{3}}X)$ or $O(\epsilon^{-\frac{1}{3}}x)$ and takes the form of an Airy function. In this regime the point of maximum amplitude has moved away from the wall and corresponds to the first turning point of $\mathrm{Ai}(X_\mathrm{T})$. As Ra_T increases through $O(1)$ values the amplitude increases smoothly through $O(\epsilon^{\frac{5}{9}})$ at $Ra_\mathrm{T} = Ra_\mathrm{TC}$ to $O(\epsilon^{\frac{1}{3}})$, changing shape smoothly from an Airy function to the second Painlevé transcendent as it does so.

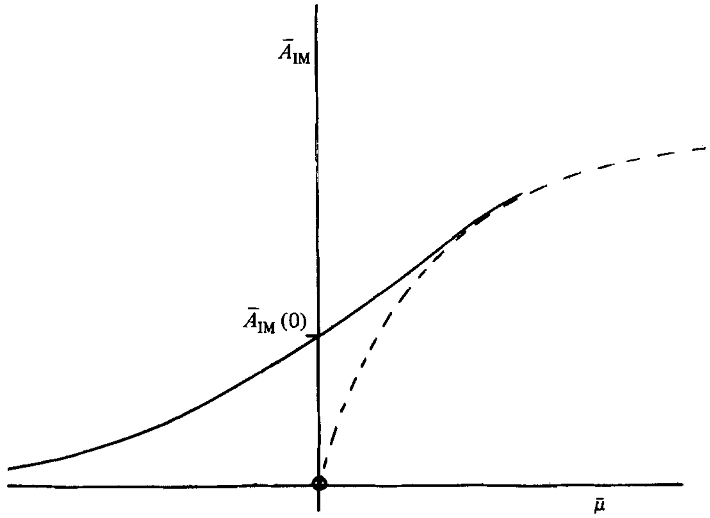

FIGURE 5. The maximum amplitude $\bar{A}_{1M}$ as a function of $\bar{\mu}$ in the parameter regime $Ra_T = Ra_{TC} + \epsilon^{\frac{4}{5}}\bar{\mu}$. Here $\bar{A}_{1M}(0) = 0.5357\,[CF'(0)/B(-Ra_{TC})]^{\frac{1}{3}}$ and the dashed curve is the asymptote $\bar{A}_{1M} = 0.5357\,[Ai'(-Ra_{TC})/B(-Ra_{TC})]^{\frac{1}{2}}\bar{\mu}^{\frac{1}{4}}$.

5. Discussion

We have shown that if the forcing at the end walls due to the incompatibility of the base flow with the endwall conditions is neglected, then the transverse mode becomes unstable when

$$Ra = \tfrac{27}{4}\pi^4(1+\epsilon^{\frac{2}{3}}2.338(-2\alpha_2^{\frac{1}{2}}F'(0))^{\frac{2}{3}}+\dots),$$

and the longitudinal mode becomes unstable when

$$Ra = \tfrac{27}{4}\pi^4(1+\epsilon^{\frac{4}{5}}3.094(-\alpha_2^{\frac{1}{4}}F'(0))^{\frac{4}{5}}+\dots).$$

We anticipate (though we have not proved) that the critical Rayleigh numbers for oblique modes (in which neither k_x or k_y is zero) lie between these values. On this basis the mode which appears first as Ra is increased for $\epsilon \ll 1$ is the longitudinal mode. These results hold whenever the horizontal boundaries are stress-free or rigid except that in the latter case $\frac{27}{4}\pi^4$ is replaced by 1707.78 and α_2 takes a value different to that given in (3.6).

The situation is more complicated when the true boundary conditions are taken into account, for, as we have shown in §4, a weak transverse mode is generated near the endwall $X = 0$ even when Ra is below critical. Its amplitude increases smoothly with Ra but is still small ($O(\epsilon)$) when the longitudinal model (for which the boundary conditions are 'perfect') becomes unstable and becomes the dominant mode. The behaviour of the solution after the onset of the longitudinal mode is beyond the scope of the present work. It may simply grow in amplitude and penetrate further into the interior of the container or it may undergo one or more further bifurcations. A true three-dimensional disturbance in the form, for example, of hexagons must not be ruled out either.

The only experimental results known to us concerning convection in a rectangular container are those undertaken by Srulijes (1979). (Rossby (1965) and Koschmieder (1966) used cylindrical containers in their experiments.) Qualitative comparisons with the present work is difficult because the experimental situation differs in many ways from our simple theoretical model. For example, the horizontal surfaces are not stress-free, nor are the sidewalls perfect insulators, and the finite extent of the apparatus in the y-direction may have important consequences on the onset of the instability. Nevertheless, a qualitative comparison between theory and experiment is quite revealing. Srulijes has observed an instability in the form of longitudinal rolls for $|\epsilon F'(0)| \gtrsim 0.2$ and $Ra > Ra_c^\infty$, but he has also noted an instability in the form of either longitudinal rolls or one transverse roll for smaller values of $|\epsilon F'(0)|$ and for subcritical values of Ra. Further, the onset of these modes occurs at smaller values of Ra as $|\epsilon F'(0)|$ increases.

The appearance of the longitudinal mode for $Ra > Ra_c$ is clearly consistent with our predictions that this is the most-unstable mode, and the appearance of the subcritical transverse mode is predicted by our analysis of §5. The amplitude of that mode is $O(\epsilon Ra F'(0))$, and if we require a threshold amplitude to be reached before this mode may be observed in an experiment we would expect the threshold value of Ra to decrease like ϵ^{-1}, and this too is consistent with Srulijes observations. What the theory cannot predict is the appearance of subcritical longitudinal rolls, and we suggest that this may be due to the finite extent of the apparatus in the y-direction, a factor which has been ignored in the present calculations. It is not possible to make an accurate comparison of our predictions of the critical value of Ra for the onset of the longitudinal mode because Srulijes does not give any information about a transition to such a mode for sufficiently small values of $|\epsilon F'(0)|$.

He does, however, give results for $Ra/Ra_c(\epsilon)$ for three values of $|\epsilon F'(0)|$ in the range 0.2 to 0.4 for the case $\epsilon = 0.1$, where $Ra_c(\epsilon)$ is the critical value of the Rayleigh number when $F'(0) = 0$. These values are some 10% below the corresponding theoretical values; but this may again be due to the finite extent of the apparatus in the y-direction, especially as this dimension was only four times the depth of the fluid layer. This difficulty is not present in the circular geometry used in Rossby's (1965) and Koschmieder's (1966) experiments, and a more accurate test of the theory may then be possible. We hope to return to this in a future paper.

We have assumed throughout that $|F'(0)|$ is $O(1)$, in which case the cold end of the container is at infinity on the lengthscale of the modulation of the amplitude. If $|F'(0)| \ll 1$, however, it is possible that this lengthscale could be as long as the horizontal dimension of the container, and the boundary condition at $X = 1$ then needs to be taken into account. For transverse rolls this requires $F'(0) \sim \epsilon^2$, and for longitudinal rolls $F'(0) \sim \epsilon^4$. The case of transverse rolls has been discussed by Daniels (1982), but the solution for longitudinal rolls, which may be more important, remains to be found.

Appendix. Asymptotic behaviour of $A_L^{iv} - (Ra_L - X_L)A_L + 2A_L^3 = 0$ as $Ra_L \to \infty$

We wish to solve this equation subject to the boundary conditions

$$A_L = A_L' = 0 \quad (X_L = 0), \quad A_L, A_L' \to 0 \quad (X_L \to \infty),$$

where $'$ denotes d/dX_L.

Numerical solutions suggest that, as $Ra_L \to \infty$, the position X_{LM} of maximum amplitude decreases, and the maximum amplitude $A_{LM} \sim [\frac{1}{2}(R_{AL} - X_{LM})]^{\frac{1}{2}}$. Following Miles' (1980) investigation of a similar second-order equation, we examine the solution in a narrow region near $X_L = 0$ by writing

$$X_L = \delta^{\frac{1}{4}}\theta, \quad \delta \ll 1.$$

Then we have
$$\delta^{-\frac{4}{5}}\frac{\mathrm{d}^4A_{\mathrm{L}}}{\mathrm{d}\theta^4}-(Ra_{\mathrm{L}}-\delta^{\frac{1}{5}}\theta)\,A_{\mathrm{L}}+2A_{\mathrm{L}}^3=0.$$

If we choose δ such that $\delta^{-\frac{4}{5}}=2Ra_{\mathrm{L}}$ and rescale A_{L} by writing $A_{\mathrm{L}}=\frac{1}{2}\delta^{-\frac{2}{5}}\bar{A}_{\mathrm{L}}$, we obtain
$$\frac{\mathrm{d}^4\bar{A}_{\mathrm{L}}}{\mathrm{d}\theta^4}=(\tfrac{1}{2}-\delta\theta)\,\bar{A}_{\mathrm{L}}-\tfrac{1}{2}\bar{A}_{\mathrm{L}}^3.$$

Solutions of this equation are now sought by expanding in powers of δ:
$$\bar{A}_{\mathrm{L}}=\bar{A}_{\mathrm{L}0}+\delta\bar{A}_{\mathrm{L}1}+\dots.$$
Leading terms then give
$$\frac{\mathrm{d}^4\bar{A}_{\mathrm{L}0}}{\mathrm{d}\theta^4}=\tfrac{1}{2}(\bar{A}_{\mathrm{L}0}-\bar{A}_{\mathrm{L}0}^3),$$
and terms $O(\delta)$ give
$$\frac{\mathrm{d}^4\bar{A}_{\mathrm{L}1}}{\mathrm{d}\theta^4}=\tfrac{1}{2}\bar{A}_{\mathrm{L}1}-\tfrac{3}{2}\bar{A}_{\mathrm{L}0}^2\bar{A}_{\mathrm{L}1}-\theta\bar{A}_{\mathrm{L}0}.$$

Miles (1980) has shown that the corresponding equations for the second order problem have simple closed-form solutions. We have not been able to find such solutions here and if $\bar{A}_{\mathrm{L}0}, \bar{A}_{\mathrm{L}1}$ need to be known for all θ, these equations need to be solved numerically. However, we can obtain some important and interesting results without detailed solutions, merely by examining the asymptotic expansions of $\bar{A}_{\mathrm{L}0}, \bar{A}_{\mathrm{L}1}$.

Let us suppose that $A_{\mathrm{L}}=A_{\mathrm{LM}}$ when $\theta=\theta_1\gg 1$. For $\theta\gg 1$ we find that
$$\bar{A}_{\mathrm{L}0}\sim 1+a_1\mathrm{e}^{\sqrt{\frac{1}{2}}\theta}\cos(\sqrt{\tfrac{1}{2}}\theta+b_1),$$
$$\bar{A}_{\mathrm{L}1}\sim -\theta,$$
where a_1, b_1 are unknown constants.

Since $A_{\mathrm{L}}=A_{\mathrm{LM}}$ at $\theta=\theta_1$ it follows that $\bar{A}'_{\mathrm{L}}=0$ at $\theta=\theta_1$, and hence
$$-a_1\mathrm{e}^{\sqrt{\frac{1}{2}}\theta_1}\cos(\sqrt{\tfrac{1}{2}}\theta_1+b_1+\tfrac{1}{4}\pi)-\delta=0.$$
These terms are of the same order of magnitude if
$$\sqrt{\tfrac{1}{2}}\theta_1\sim -\ln\delta\quad(\delta\to 0).$$
Hence
$$X_{\mathrm{LM}}=\delta^{\frac{1}{5}}\theta_1\sim -\sqrt{2}\delta^{\frac{1}{5}}\ln\delta\quad(\delta\to 0),$$
$$\sim 2^{\frac{1}{4}}Ra_{\mathrm{L}}^{-\frac{1}{4}}\ln Ra_{\mathrm{L}}\quad(Ra_{\mathrm{L}}\to\infty).$$
Also
$$A_{\mathrm{LM}}\sim\tfrac{1}{2}\delta^{-\frac{2}{5}}\quad(\delta\to 0),$$
$$\sim(\tfrac{1}{2}Ra_{\mathrm{L}})^{\frac{1}{2}}\quad(Ra_{\mathrm{L}}\to\infty),$$
as anticipated by the numerical computations.

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